COMBINATORICA 6 (1) (1986) 23—27

A PARITY DIGRAPH HAS A KERNEL

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Received 5 April 1984

We show that every digraph has a kernel (i.e. an absorbing and independent set) under the following parity condition: For every pair of vertices x, y $x \neq y$ all minimal directed paths between x and y have the same length parity.

1. Introduction

A kernel of a digraph $D=(V, U)$ is a subset $K \subseteq V$ which is both independent and absorbing. This notion was introduced after game theory concept due to Von-Neumann—Morgenstern [1], [7], [8].

Not all digraphs have a kernel, different sufficient conditions are known in the literature implying the existence of kernels [2], [3], [4], [5], [6].

In this paper, answering a problem proposed by P. Duchet, we prove that a parity digraph has a kernel.

2. Definitions and notation

A digraph $D=(V, U)$ is a finite directed graph without loops nor multiple arcs. V is the vertex-set of D and U its arc set. We have $U \subseteq \{(V \times V) - \{(v, v), v \in V\}\}$. By a path, we mean an elementary path.

A *path* $C(v_1, v_p)=(v_1, \ldots, v_p)$ of a digraph D is *minimal* when (v_i, v_j) is not an arc of D for every i, j $2 \leqq i+1 < j \leqq p$.

The *parity of a path* C is the parity of the number of its arcs.

A digraph $D=(V, U)$ is called a *parity digraph* if, for every pair of vertices x and y $(x \neq y)$ of D, all minimal paths between x and y have the same parity.

For a digraph $D=(V, U)$ and $S \subseteq V$ we denote by $G[S]$ the subdigraph

AMS subject classification (1980): 05 C 20

induced by S. We set

$$\Gamma^+(x) = \{y \in V | (x, y) \in U\}$$

$$\Gamma^-(x) = \{y \in V | (y, x) \in U\}$$

$$\Gamma^+(A) = \bigcup_{x \in A} \Gamma^+(x)$$

$$\Gamma^-(A) = \bigcup_{x \in A} \Gamma^-(x).$$

We recall the following definitions [1].

An *absorbing set* of D is a subset K if V such that $\Gamma^-(K) \cup K = V$.

An *independent set* of D is a subset K of V such that $(\Gamma^-(K) \cup \Gamma^+(K)) \cap K = \emptyset$. D is said to be *kernel perfect* if every induced subdigraph has a kernel. A vertex y is a *successor* (resp. *predecessor*) of a vertex x if $y \in \Gamma^+(x)$ (resp. $y \in \Gamma^-(x)$).

We denote by

$C(x_0, x_{p+1}) = (x_0, x_1, \ldots, x_{p+1})$ a path joining x_0 to x_{p+1}, with $(x_i, x_{i+1}) \in U$ for $0 \leqq i < p$.

$C(a, b)$ the subpath of $C(x_0, x_{p+1})$ joining $a = x_i$ to $b = x_j$, $0 \leqq i \leqq j \leqq p+1$.

$C(x, y) \,\&\, C(y, z)$ the path from x to z obtained by prolongation of the path $C(x, y)$ by the path $C(y, z)$.

3. The main result

Theorem. *A parity digraph has a kernel.*

Proof. The proof is by induction on the number of vertices of $D = (V, U)$, D a parity graph.

The theorem is true for digraphs having less than four vertices.

Let x_0 be a vertex of D. By hypothesis $D \setminus x_0$ has a kernel N. We define a sequence of sets associated to x_0 as follows (see Figure 1).

$$B_0 = \Gamma^+(x_0)$$

$$N_0 = \Gamma^-(B_0) \cap N$$

$$B_1 = \Gamma^+(N_0) \cap (V \setminus x_0 \setminus B_0)$$

$$N_1 = \Gamma^+(B_1) \cap (N \setminus N_0)$$

$$\vdots$$

$$B_i = \Gamma^+(N_{i-1}) \cap \Big(V \setminus x_0 \setminus \bigcup_{j=0}^{i-1} B_j\Big)$$

$$N_i = \Gamma^+(B_i) \cap \Big(N \setminus \bigcup_{j=0}^{i-1} N_j\Big)$$

$$\vdots$$

Let k be the smallest integer such that $N_{k+1} = \emptyset$. It is clear that if x_0 has a successor in N, then N is a kernel of D. Thus $\Gamma^+(x_0) \cap N = \emptyset$.

We break the proof into several steps.

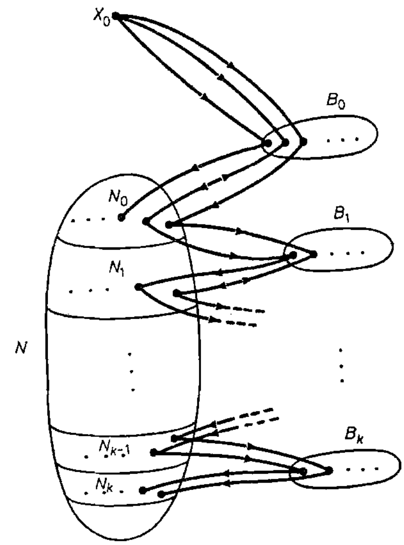

Fig. 1

(1) *In the sequence defined above B_0 is not empty.*

Otherwise, let N' be a kernel in the subdigraph $G[V\setminus\Gamma^-(x_0)]$, clearly $N''=\{x_0\}\cup N'$ is a kernel in the digraph D. (A kernel in $G[V\setminus\Gamma^-(x_0)]$ exists by hypothesis of induction).

(2) If $N_1=\emptyset$, i.e. every vertex adjacent to N_0 in $D\setminus x_0$ has a successor in N_0 we take a kernel N' in the subdigraph $G[V\setminus(N_0\cup\Gamma^-(N_0))]$. (It exists by the hypothesis of induction).

We show that the set $N''=N_0\cup N'$ is a kernel of D.

— It is absorbing by construction.
— It is independent since x_0 is not joined to vertices of N_0, otherwise we obtain a contradiction with the parity condition.

(3) The subdigraph $G[V\setminus\bigcup_{i=0}^{k}(N_i\cap\Gamma^-(N_i))]$ has a kernel N' by the hypothesis of induction. *We show that the set $N''=(\bigcup_{i=0}^{k}N_i)\cup N'$ is a kernel of the digraph* D. It is absorbing by construction.

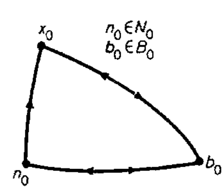

Fig. 2

N'' is independent for the following reasons:

a) N' and $\bigcup_{i=0}^{k} N_i$ are independent subsets by definition.

b) No vertex of $N' \setminus x_0$ is joined to vertex of N_i, $i=0, \ldots, k$, because otherwise it would not be in $G[V \setminus \bigcup_{i=0}^{k} (N_i \cup \Gamma^-(N_i))]$.

c) It remains to prove the following step.

(4) x_0 *is not joined to vertices* of N_i, $i=0, \ldots, k$.

The proof is by induction on the N_i. x_0 cannot be joined to N_0, otherwise we obtain a contradiction with the parity condition.

Suppose that x_0 is not joined to N_i, $0 \leqq i \leqq p$ and show that it is true for N_{p+1}. Suppose that on the contrary x_0 is joined to $n_{p+1} \in N_{p+1}$. By construction, there exists a path from x_0 to n_{p+1}: $C(x_0, x_{p+1})$ alternately going through vertices of B_i and N_i $0 \leqq i \leqq p+1$. We show that the only minimal path from x_0 to n_{p+1} induced by the vertices of $C(x_0, n_{p+1})$ is $C(x_0, n_{p+1})$. Thus, we obtain a contradiction with the parity of digraph D. (Because, from x_0 to n_{p+1} the minimal path has an even parity but from n_{p+1} to x_0 we have an arc $(n_{p+1}, x_0) \in U$.) For that, we must prove there are no arcs (b_i, b_j) when $b_i \in B_i$ and $b_j \in B_j$ for all $0 \leqq i < j \leqq p$ of the path $C(x_0, n_{p+1})$ (see Figure 3).

We observe that there are no arcs of the following forms:

(n_i, b_j) $\quad i < j-1 \quad$ by definition of B_{i+1},

(b_j, n_i) $\quad i > j \quad$ by definition of N_i.

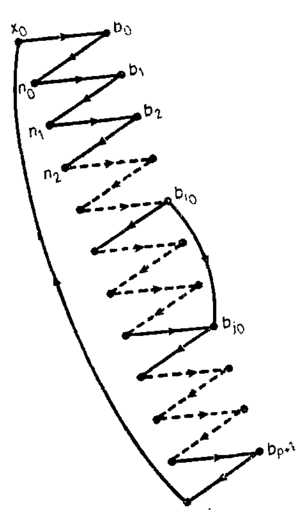

Fig. 3

We suppose that (b_i, b_j) exists $i<j$. Set

$$j_0 = \min\big(j|(b_i, b_j) \quad \text{exists} \quad i<j\big)$$

$$i_0 = \min\big(i|(b_i, b_{j_0}) \quad \text{exist} \quad i<j_0\big).$$

Then (b_{i_0}, b_{j_0}) is the first arc met by the path $C(x_0, n_{p+1})$. Since there are no arcs (b_i, b_j) before (b_{i_0}, b_{j_0}), a minimal path induced by the path $C(x_0, n_{j_0-1})$ is exactly $C(x_0, n_{j_0-1})$ and has an even parity. On the other hand there exists a minimal path induced by $C(n_{j_0-1}, n_{p+1})\,\&\,(n_{p+1}, x_0)$ n_{j_0-1} to x_0 with an even parity which must go through b_{j_0} (because by construction of B_{j_0} the arcs (n_{j_0-1}, b_j), $j>j_0$ cannot exist).

Then, a minimal path induced by $C(b_{j_0}, n_{p+1})\,\&\,C(n_{p+1}, x_0)$ from b_{j_0} to x_0 has an odd parity but a minimal path induced by $C(x_0, b_{i_0})\,\&\,C(b_{i_0}, b_{j_0})$ has an even parity (since $C(x_0, b_{i_0})$ is with an odd parity and there are no arcs (b_i, b_j) before (b_{i_0}, b_{j_0})). We obtain a contradiction with the parity condition of digraph. Hence (b_i, b_j) for all $i<j$ cannot exist. ▮

Corollary. *Parity digraph are kernel perfect.*

Proof. Obvious, since every minimal path in an induced subdigraph of a digraph D is also a minimal path in D.

4. Remarks

The ideas used in the first (constructive) part of the proof are closely related to a construction of V. Neumann Lara [4].

On the other hand, we mention the forthcoming paper [2] where we prove a similar result for undirected parity graphs with an orientation condition.

Acknowledgement. Finally, I wish to thank P. Duchet, M. Las Vergnas and H. Meyniel for their suggestions that improved the presentation of this paper.

References

[1] C. Berge, *Graphes et Hypergraphes*, Dunot 1970.
[2] M. Blidia, Kernels in parity graphs with an orientation condition, to appear.
[3] P. Duchet, Graphes noyaux parfaits, *Ann. Disc. Math.* 9 (1980) 93—102.
[4] H. Galeana-Sanchez, A theorem about a conjecture of H. Meyniel on kernel perfect graph, *Universidad Nacional A de Mexico.*
[5] H. Meyniel, Contribution à l'étude de quelques problèmes en théorie des graphes (Circuits hamiltoniens, coloration, noyaux), *Thèse Paris VI* (1982).
[6] V. Neumann-Lara, Seminucleos de uno digrafica, *Anales del Instituto de Matematicas* **11** (1971), *Universidad Nacional A de Mexico.*
[7] J. von Neumann and O. Morgenstern, *Theory of games and economic behavior*, Princeton University Press, Princeton 1944.
[8] M. Richardson, Solutions of irreflexive relations, *Annals of Math.* **58** (1953), 573—580.

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